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THE NATURAL FREQUENCY OF RECTANGULAR SYMMETRIC ANGLE-PLY LAMINATES RESTING ON AN ELASTIC FOUNDATION

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1. INTRODUCTION

Research focused on the mechanical response, bending, vibration and buckling, of symmetric composite angle-ply laminates is quite extensive [1]. Research into the response of composite laminates resting on an elastic foundation include the work of Shen *et al.* [2] who examined the thermomechanical buckling analysis of laminates. A two parameter foundation model is used and the analysis is accomplished using a perturbation technique. Xu and Chia [3] incorporated a non-linear shear deformable theory to study the vibration of thick circular composite plates. Several foundation models are used including a non-linear Winkler model. A hybrid Fourier–Bessel series was incorporated in the solution. Chen and Gurdal [4] examined the three dimensional stress distribution, created by a transverse load, of infinite orthotropic plates. The authors used Fourier transforms to solve the problem. Tomar *et al.* [5] computed the natural frequencies of circular, non-uniform isotropic plates using the method of Frobeius. Raju and Rao [6] investigated the interaction between buckling and vibration of rectangular, orthotropic plates resting on an elastic foundation using a Winkler model. The problem is formulated using the principle of total potential energy and uses the Rayleigh–Ritz method to compute the fundamental frequency. Fadhil and El-Zafrany [7] examined thick Reissner plates using boundary element analysis. Both one and two parameter foundation models were studied.

2. PROBLEM STATEMENT

The equation governing the behavior of the symmetric angle-ply laminate resting on an elastic foundation using a Winkler model is given by

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3\,\partial y} + 2D_k\frac{\partial^4 w}{\partial x^2\,\partial y^2} + 4D_{26}\frac{\partial^4 w}{\partial x\,\partial y^3} + D_{22}\frac{\partial^4 w}{\partial y^4} + k_f w = \rho\omega^2 w. \qquad (1)$$

In equation (1) w is the mode shape, ρ is the mass density, ω is the frequency, and x and y represent spatial co-ordinates. In addition, D_{ij} are the usual flexural stiffness parameters, with $D_k = D_{12} + 2D_{66}$, and k_f represents the stiffness of the foundation. Simply supported boundary conditions given by

$$\text{on } x = 0, a, \quad w = 0,\ M_x = 0; \qquad \text{on } y = 0, b, \quad w = 0,\ M_y = 0, \qquad (2)$$

are used for this problem.

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3. PROBLEM FORMULATION

A discrete set of equations corresponding to equation (1) is obtained using the Ritz method. To this end, the displacement $w(x, y)$ is expanded in a complete, kinematically admissible basis given by

$$w(x, y) = \sum_m \sum_n w_{mn} \psi_{mn}(x, y), \tag{3}$$

where w_{mn} are constants and

$$\psi_{mn}(x, y) = \rho X_m(x/a) Y_n(y/b). \tag{4}$$

Here X_m and Y_n are basis functions to be selected. Substituting equations (3) and (4) into equation (1), and taking the L_2 inner product with ψ_{qp}, provides

$$[\mathbf{K}]\{\boldsymbol{\alpha}_{mn}\} = \lambda_{mn}[\mathbf{M}]\{\boldsymbol{\alpha}_{mn}\}. \tag{5}$$

Here, $\{\boldsymbol{\alpha}_{mn}\}$ is interpreted as the eigenvector with corresponding eigenvalue λ_{mn}, $[\mathbf{K}]$ is the stiffness matrix and $[\mathbf{M}]$ is the mass matrix. The elements of these matrices are given by

$$\begin{aligned}(a^4/ab)K_{pqmn} = {} & D_{11}A_{pm}b_{qn} + D_{12}C_{pm}c_{nq}R^2 + 2D_{16}H_{pm}g_{nq}R + D_{12}C_{mp}c_{qn}R^2 \\ & + D_{22}B_{pm}a_{qn}R^4 + 2D_{26}G_{mp}h_{qn}R^3 + 2D_{16}H_{mp}g_{qn}R + 2D_{26}G_{pm}h_{nq}R^3 \\ & + 4D_{66}E_{pm}e_{qn}R^2 + a^4k_f B_{pm}b_{qn},\end{aligned} \tag{6}$$

where R is the aspect ratio given as a/b and

$$(1/ab)M_{pqmn} = \rho B_{pm}b_{qn}. \tag{7}$$

In equations (6) and (7), the following definitions have been introduced

$$\begin{gathered} A_{pm} = (X''_p, X''_m), \qquad a_{qn} = (Y''_q, Y''_n), \qquad B_{pm} = (X_p, X_m), \qquad b_{qn} = (Y_q, Y_n), \\ C_{pm} = (X''_p, X_m), \qquad c_{qn} = (Y''_q, Y_n), \\ E_{pm} = (X'_p, X'_m), \qquad e_{qn} = (Y'_q, Y'_n), \qquad G_{pm} = (X'_p, X_m), \qquad g_{qn} = (Y'_q, Y_n), \\ H_{pm} = (X''_p, X'_m), \qquad h_{qn} = (Y''_q, X'_n). \end{gathered} \tag{8}$$

Symbolically, $()'$ represents the derivative with respect to the basis argument and $(\bullet, \bullet)$ represents the L_2 inner product on [0, 1].

The above equations for K_{pqmn} and M_{pqmn} are quite general. They are independent of the particular set of basis functions, although the matrices (8) depend upon the basis selected. Therefore they can be used if X_p and Y_q are kinematically admissible polynomials, beam shape functions, or any other set of kinematically admissible functions. Here the basis functions for the composite laminate will be the beam shape functions for a similarly supported beam.

4. SENSITIVITY ANALYSIS

An approximate expression for the eigenvalue λ_{mn} can be determined by introducing parameters S_1 and S_2 into equation (5) and considering

$$[\hat{\mathbf{K}}(S_1)]\{\hat{\boldsymbol{\alpha}}_{mn}(S_1, S_2)\} = \hat{\lambda}_{mn}(S_1, S_2)[\hat{\mathbf{M}}(S_2)]\{\hat{\boldsymbol{\alpha}}_{mn}(S_1, S_2)\}, \tag{9}$$

where

$$[\hat{\mathbf{K}}(S_1)] = [\mathbf{K}_D] + S_1[\boldsymbol{\Delta}\mathbf{K}], \qquad [\hat{\mathbf{M}}(S_2)] = [\mathbf{M}_D] + S_2[\boldsymbol{\Delta}\mathbf{M}]. \tag{10}$$

Here, $[\mathbf{K}_D]$ and $[\mathbf{M}_D]$ are diagonal matrices obtained from $[\mathbf{K}]$ and $[\mathbf{M}]$, respectively, by deleting all off-diagonal elements; $[\mathbf{\Delta K}]$ and $[\mathbf{\Delta M}]$ are matrices which have zeros on the diagonal and contain only the off-diagonal elements of $[\mathbf{K}]$ and $[\mathbf{M}]$. The parameters S_1 and S_2 range from 0 to 1. If $S_1 = S_2 = 0$, the solution to equation (9) becomes the ratio of the diagonal elements of the stiffness matrix $[\mathbf{K}_D]$ and mass matrix $[\mathbf{M}_D]$. If $S_1 = S_2 = 1$, the original eigenvalue problem, equation (5), is recovered. The desired eigenvalue λ_{mn} is obtained by expanding $\hat{\lambda}_{mn}$ in a Maclaurin series about $S_1 = S_2 = 0$ and evaluating at $S_1 = S_2 = 1$. Thus

$$\lambda_{mn} = \hat{\lambda}_{mn}(1, 1) \cong \hat{\lambda}_{mn}(0, 0) + \delta\hat{\lambda}_{mn}(0, 0) + \tfrac{1}{2}\delta^2\hat{\lambda}_{mn}(0, 0). \tag{11}$$

The desired results appearing on the right side of equation (11) can be shown to be

$$\hat{\lambda}_{mn}(0, 0) = K_{mnmn}/M_{mnmn}, \qquad \delta\hat{\lambda}_{mn}(0, 0) = 0 \tag{12}$$

and

$$\delta^2\hat{\lambda}_{mn} = -\frac{2}{M^2_{mnmn}} \sum_{p \neq m} \sum_{q \neq n} \left\{ \frac{[K_{mnmn}\Delta M_{pqmn} - M_{mnmn}\Delta K_{pqmn}]^2}{K_{mnmn}M_{pqpq} - K_{pqpq}M_{mnmn}} \right\}. \tag{13}$$

Indeed, Barton and Reiss [8] have provided a complete derivation for these terms by considering the buckling of a symmetric angle-ply laminate. Substituting equations (12) and (13) into equation (11) provides the required quadratic approximate closed form expression of

$$\lambda_{mn} = \frac{K_{mnmn}}{M_{mnmn}} - \frac{1}{M^2_{mnmn}} \sum_{p \neq m} \sum_{q \neq n} \left\{ \frac{[K_{mnmn}\Delta M_{pqmn} - M_{mnmn}\Delta K_{pqmn}]^2}{K_{pqpq}M_{mnmn} - K_{mnmn}M_{pqpq}} \right\}. \tag{14}$$

5. DISCUSSION AND RESULTS

Equation (14) can be specialized for the problem at hand by selecting a set of basis functions, evaluating the matrices appearing in equation (8), and then evaluating the stiffness and mass matrices. A final substitution of the stiffness and mass matrices into equation (14) provides the desired expression. A set of normalized beam shape functions

$$X_m(x/a) = \sqrt{2}\sin(m\pi x/a), \qquad Y_n(y/b) = \sqrt{2}\sin(n\pi y/b), \tag{15}$$

were selected as the set of basis functions. Utilizing this set of basis functions and evaluating the matrices appearing in equation (8), provides

$$A_{pm} = \mu_p^4\delta_{pm}, \qquad a_{qn} = \nu_n^4\delta_{qn}, \qquad B_{pm} = \delta_{pm}, \qquad b_{qn} = \delta_{qn}, \qquad C_{pm} = -p^2\pi^2\delta_{pm},$$

$$c_{qn} = -q^2\pi^2\delta_{qn}, \qquad E_{pm} = -C_{pm}, \qquad e_{qn} = -c_{qn},$$

$$H_{pm} = -p^2 m\pi^3\phi_{pm}, \qquad h_{qn} = -q^2 n\pi^3\phi_{qn}, \qquad G_{pm} = p\pi\phi_{pm}, \qquad g_{qn} = q\pi\phi_{qn}, \tag{16}$$

where

$$\phi_{\alpha\beta} = \left\{ \begin{matrix} 2\alpha/\pi(\alpha^2 - \beta^2) & \alpha + \beta, & \text{odd} \\ 0 & \alpha + \beta, & \text{even} \end{matrix} \right\}. \tag{17}$$

Here, μ_m^2 and v_n^2 are frequencies corresponding to the simply supported beam, and δ_{ij} is the Kronecker delta. The stiffness and mass matrices appearing in equations (6) and (7) can be determined and become

$$\begin{aligned}(a^4/ab)K_{pqmn} = {} & D_{11}\mu_P^4\delta_{pm}\delta_{qn} + 2D_k p^2q^2\pi^4\delta_{mp}\delta_{qn}R^2 + D_{22}v_q^4R^4\delta_{pm}\delta_{qn} \\ & - 2D_{16}pm\pi^4(pn\phi_{pm}\phi_{qn} + mq\phi_{mp}\phi_{qn})R \\ & - 2D_{26}qn\pi^4(np\phi_{pm}\phi_{nq} + qm\phi_{mp}\phi_{qn})R^3 + a^4k_f\delta_{pm}\delta_{qn},\end{aligned} \tag{18}$$

and

$$(1/ab)M_{pqmn} = \rho\,\delta_{pm}\delta_{qn}. \tag{19}$$

An explicit result for the fundamental frequency is obtained by substituting equations (18) and (19) into equation (14) with $m = n = 1$. Doing so provides

$$\begin{aligned}a^4\rho\lambda_{11} = {} & \pi^4[D_{11} + 2D_kR^2 + D_{22}R^4 + a^4k_f/\pi^4] - 128R\pi^4\sum_{p\neq 1}^{N}\sum_{q\neq 1}^{N}\frac{p^2q^2}{(p^2-1)^2(q^2-1)^2} \\ & \times\frac{[D_{16}(p^2+1) + D_{26}(q^2+1)R^2]^2}{[D_{11}(p^4-1) + 2D_k(p^2q^2-1)R^2 + D_{22}(q^4-1)R^4]}.\end{aligned} \tag{20}$$

Equation (20) is the desired result representing the fundamental frequency of a rectangular symmetric laminate resting on an elastic foundation. To aid in the computation of the frequency, the following normalized fundamental frequency k_{11} is introduced:

$$k_{11} = \sqrt{\lambda_{11}} = a^2\omega_{11}\sqrt{\rho/(U_1t^3)}, \tag{21}$$

where U_1 is a material invariant property and t is the laminate's thickness. Comparison with the normalized fundamental frequency from the Rayleigh–Ritz method, identified as k, provides the means of assessing the accuracy of the quadratic approximation.

The laminate studied is a four ply, symmetric laminate $[\theta/-\theta]_s$ with the following material stiffness ratios corresponding to graphite–epoxy: $E_{11}/E_{22} = 40{\cdot}0$, $E_{11}/G_{12} = 80{\cdot}0$, $v_{12} = 0{\cdot}30$. Finally, nine terms, $N = 0$, were taken in the displacement expansion (3).

Tabulated numerical results are presented for results $k_f = 0{\cdot}0$, corresponding to no foundation support, for $k_f = 10^{10}$ lb/in^3 representing a moderate foundation stiffness, and for $k_f = 10^{12}$ lb/in^3, representing a stiff foundation. For each foundation response, results are presented for $R = 1$, 2 and 5 and various ply orientation angles, in increments of 15°, starting with $\theta = 0°$. From these tables one is able to assess the accuracy of the quadratic approximate formula, equation (20), as well as determine the effect of foundation stiffness on the fundamental frequency.

Table 1 provides results for the laminate without any soil interaction, $k_f = 0{\cdot}0$. For the square laminate, $R = 1$, the maximum percent difference is 1·58% at $\theta = 45°$, corresponding to the largest values of the coupling stiffness. Here, the Rayleigh–Ritz predicts $k_1 = 46{\cdot}77$ and the approximate expression predicts $k_{11} = 47{\cdot}51$. For $R = 2$, the largest discrepancy again occurs at $\theta = 45°$ with a value of 1·33%. A shift in the location from $\theta = 45°$ to $\theta = 30°$ of largest error results if one sets $R = 5$. Here, the error is 0.47%.

Table 2 provides information in which the effect of foundation response is noticed. A small increase in the fundamental frequency results in an increase of the foundation stiffness. For instance, the square orthotropic laminate, $\theta = 0°$, experiences a 0·32% increase in frequency to 38·10 and for the $\theta = 45°$, an increase of 0·19% results. The maximum percent difference occurs at $\theta = 45°$ for $R = 1$ and 2 and $\theta = 30°$ for $R = 5$. In addition, there is a modest decrease in these values from the previous case.

Table 1

Variation of normalized fundamental frequency with ply orientation for rectangular laminate and foundation stiffness $k_f = 0{\cdot}0$.

θ (°)	R	Rayleigh–Ritz (k)	Approximate Formula (k_{11})	% Difference
0	1	37·98	37·98	0·00
15	1	39·83	40·01	−0·45
30	1	44·33	44·95	−1·40
45	1	46·77	47·51	−1·58
60	1	44·33	44·95	−1·40
75	1	39·83	40·01	−0·45
90	1	37·98	37·98	0·00
0	2	11·72	11·72	0·00
15	2	14·96	15·04	−0·53
30	2	21·24	21·49	−1·18
45	2	27·02	27·38	−1·33
60	2	31·80	32·04	−0·75
75	2	35·34	35·38	−0·11
90	2	36·70	36·70	0·00
0	5	6·22	6·22	0·00
15	5	7·87	7·88	−0·13
30	5	12·74	12·80	−0·47
45	5	20·25	20·32	−0·35
60	5	28·24	28·27	−0·11
75	5	34·22	34·23	−0·03
90	5	36·43	36·43	0·00

Table 2

Variation of normalized fundamental frequency with ply orientation for rectangular laminate and foundation stiffness $k_f = 1 \times 10^{10}$

θ (°)	R	Rayleigh–Ritz (k)	Approximate Formula (k_{11})	% Difference
0	1	38·1	38·1	0·00
15	1	39·94	40·12	−0·45
30	1	44·43	45·05	−1·40
45	1	46·86	47·6	−1·58
60	1	44·43	45·05	−1·40
75	1	39·94	40·12	−0·45
90	1	38·1	38·1	−0·00
0	2	12·08	12·08	−0·00
15	2	15·25	15·25	−0·00
30	2	21·44	21·7	−1·21
45	2	27·18	27·54	−1·32
60	2	31·94	32·18	−0·75
75	2	35·46	35·5	−0·11
90	2	36·82	36·82	0·00
0	5	6·88	6·88	0·00
15	5	8·41	8·41	0·00
30	5	13·07	13·13	−0·46
45	5	20·47	20·53	−0·29
60	5	28·4	28·43	−0·11
75	5	34·35	34·36	−0·03
90	5	36·55	36·55	0·00

TABLE 3

Variation of normalized fundamental frequency with ply orientation for rectangular laminate and foundation stiffness $k_f = 1 \times 10^{12}$

θ (°)	R	Rayleigh–Ritz (k)	Approximate Formula (k_{11})	% Difference
0	1	48·09	48·09	0·00
15	1	49·56	49·71	−0·30
30	1	53·25	53·76	−0·96
45	1	55·29	55·92	−1·14
60	1	53·25	53·76	−0·96
75	1	49·56	49·71	−0·30
90	1	48·09	48·09	0·00
0	2	31·74	31·74	0·00
15	2	33·07	33·11	−0·12
30	2	36·35	36·5	−0·41
45	2	40	40·25	−0·65
60	2	43·37	43·55	−0·42
75	2	46·03	46·06	−0·07
90	2	47·09	47·09	0·00
0	5	30·14	30·14	0·00
15	5	30·53	30·53	0·00
30	5	32·13	32·15	−0·06
45	5	35·78	35·82	−0·11
60	5	40·84	40·86	−0·05
75	5	45·18	45·18	0·00
90	5	46·87	46·87	0·00

Finally, Table 3 provides the data for the last foundation stiffness. Here, a more dramatic increase in the fundamental frequency results. Again considering the square orthotropic laminate, an increase of 26·22% to 48·09 is noticed and for $\theta = 45°$, a 17·98% increase to 55·29 results. Moreover, an improvement in accuracy of 0·44% is noticed. An increase in accuracy is noticed for the other laminate configurations.

6. CONCLUSION

In this paper an approximate closed-form expression was developed and used to predict the fundamental frequency of symmetric laminated composite plates resting on an elastic foundation. Results were presented for various foundation stiffnesses, laminate aspect ratios, and ply orientation. A very good, if not excellent, comparison exists when compared with the Rayleigh–Ritz method.

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